RENORMALIZATION OF SELF-CONSISTENT Φ-DERIVABLE APPROXIMATIONS*

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Within finite temperature field theory, we show that truncated non-perturbative self-consistent Dyson resummation schemes can be renormalized with local vacuum counterterms. For this the theory has to be renormalizable in the usual sense and the self-consistent scheme must follow Baym's Φ-derivable concept. Our BPHZ-renormalization scheme leads to renormalized self-consistent equations of motion. At the same time the corresponding 2PI-generating functional and the thermodynamic potential can be renormalized with the same counterterms used for the equations of motion. This guarantees the standard Φ-derivable properties like thermodynamic consistency and exact conservation laws also for the renormalized approximation schemes. We give also a short overview over symmetry properties of the various functions defined within the 2PI scheme for the case that the underlying classical field theory has a global linearly realized symmetry.

1. Introduction

Describing hot and dense systems of strongly interacting particles, one is led to the use of *dressed propagators* within non-perturbative Dyson-resummation schemes. Especially this becomes unavoidable if *damping width effects* are significant for the physical situation in question.

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Based on functional formulations by Luttinger and Ward[1] and Lee and Yang[2] Baym and Kadanoff[3] studied a special class of self-consistent Dyson approximations, which later was reformulated in terms of a variational principle, defining the so-called Φ-*derivable approximations*[4]. The variational principle, applied to approximations of the Φ-functional, leads to closed coupled equations of motion for the mean field and the propagator, which guarantee the exact conservation of the expectation values of conserved currents and thermodynamical consistency, since at the same time the approximated Φ-functional is an approximation of the thermodynamic potential.

Later this concept was generalized to the relativistic case and rederived within the path integral formalism by Cornwall, Jackiw, and Tomboulis[5]. It is no formal problem to extend this formulation to the general Schwinger-Keldysh real-time contour[6,7] and thus to generalize the concept to non-equilibrium problems.

Here we discuss the problem, how to *renormalize* the equations of motion, derived from Φ-derivable approximations for relativistic quantum field theories. Generalizing the work of Bielajew and Serot[8,9] we show that any Φ-derivable approximation of a perturbatively renormalizable theory is also renormalizable in the usual sense. Further we prove that at finite temperature only *temperature-independent counterterms* are necessary to give finite equations of motion[10]. The counterterms can be interpreted as renormalization of the wave function and the vacuum parameters of the quantum field theory, like the particle masses and the coupling constants.

Further we demonstrate the possibility to treat numerically Φ-derivable approximations with generic two-point contributions to the self-energy beyond pure gap-equation approximations with "tadpole self-energies") giving rise to a finite in-medium damping width of the involved particles[11].

Another important question is whether the approximations respect underlying symmetries of the classical action functional. Contrary to perturbation theory in general the solution of the Φ-derivable equations of motion violates the Ward-Takahashi identities of symmetries. This was discussed first by Baym and Grinstein[12] on the example of the $\mathrm{O}(N)$-symmetric linear σ-model. The reason can be traced back to a violation of *crossing symmetry* for approximations of the Φ-functional: The solution of the Φ-derivable equations of motion is equivalent to a resummation of the self-energy in any order of the expansion parameter of the Φ-functional approximation. This involves intrinsically the resummation of higher vertex functions, which is incomplete, because certain channels are missing, being taken into account only by approximations of the Φ functional at higher orders.

We show that this can partially be cured by defining a non-perturbative approximation to the usual effective quantum action. This admits one to define vertex-functions which fulfill the Ward-Takahashi identities of the symmetry. These crossing symmetric vertex-functions are defined by equations of motion which solution is equivalent to a further resummation of the channels, missing intrinsically in the Φ-derivable self-energy resummation[13].

2. The 2PI generating functional

We start with the defining path integral for the two-particle (2PI) irreducible quantum action for the state of thermal equilibrium. For this case we use the Schwinger-Keldysh closed real-time path extended by an imaginary part making use of the fact that the unnormalized thermal density operator $\exp(-\beta \mathbf{H})$, with β denoting the inverse temperature of the system and $\mathbf{H}$ its Hamilton operator, can be included within the path integral as a time evolution parallel to the imaginary axis (see Fig. 1). We consider the

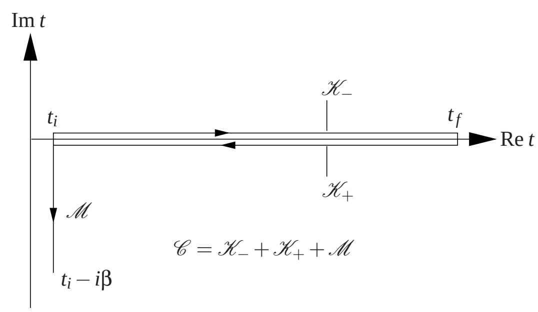


Figure 1. The Schwinger-Keldysh closed time path modified for the application to thermal equilibrium quantum field theory.

local relativistic renormalizable quantum field theory for one scalar field ϕ with the dynamics defined by the classical action

$$S[\phi] = \int_{\mathcal{C}} \mathrm{d}(1) \left[\frac{1}{2} (\partial_\mu \phi_1)(\partial^\mu \phi_1) - \frac{m^2}{2} \phi_1^2 - \frac{\lambda}{4!} \phi_1^4 \right]. \tag{1}$$

Here and in the following $\int_{\mathcal{C}} \mathrm{d}(123\ldots) f_{123\ldots}$ denotes an integral over a function f of space-time arguments $x_1, x_2, \ldots$. The time variable is assumed to be defined along the contour depicted in Fig. 1.

The generating functional is given by

$$\Gamma[\varphi, G] = S[\varphi] + \frac{\mathrm{i}}{2} \operatorname{Tr}(M^2 G^{-1}) + \frac{\mathrm{i}}{2} \int_{\mathcal{C}} \mathrm{d}(12) D_{12}^{-1} (G_{12} - D_{12}) + \Phi[\varphi, G] \tag{2}$$

with

$$D_{12}^{-1} = \frac{\delta^2 S[\varphi]}{\delta\varphi_1 \delta\varphi_2}. \tag{3}$$

In terms of diagrams the functional Φ consists of closed two-loop diagrams, which are built with lines representing *exact propagators* G and point-vertices with respect to the field ϕ derived from the action $S[\varphi + \phi]$.

The equations of motion are determined by the stationary point of the functional (2):

$$\frac{\delta\Gamma}{\delta\varphi_1} \stackrel{!}{=} 0, \quad \frac{\delta\Gamma}{\delta G_{12}} \stackrel{!}{=} 0. \tag{4}$$

Using (2) these equations of motion read

$$\frac{\delta S}{\delta\varphi_1} = -\frac{\mathrm{i}}{2}\int_{\mathcal{C}} \mathrm{d}(1'2') \frac{\delta D_{1'2'}^{-1}}{\delta\varphi_1} G_{1'2'} - \frac{\delta\Phi}{\delta\varphi_1}, \tag{5}$$

$$\Sigma_{12} := D_{12}^{-1} - G_{12}^{-1} = 2\mathrm{i}\frac{\delta\Phi}{\delta G_{12}}. \tag{6}$$

From the latter equation we see that the derivative of Φ with respect to G gives the *exact self-energy* of the theory at presence of a mean field φ, which in turn is determined from (5) self-consistently. Since lines in diagrams contributing to an expansion of Φ stand for exact Green's functions G all these lines must not contain any self-energy insertions, i.e., the self-energy is represented as the sum of all *skeleton diagrams*. Since the derivative of a diagram with respect to G means to open any line contained in it and then adding all the so obtained diagrams, the functional Φ consists of all *closed two-particle irreducible diagrams* with at least two loops.

A Φ-derivable *approximation* is defined as the truncation of the functional Φ to a finite (e.g., coupling-constant or $\hbar$-expansion) or an explicitly resummable infinite subset of 2PI diagrams. The mean field φ and Green's function G are then determined by the self-consistent closed equations of motion (5-6).

3. Renormalization of ϕ^4-theory

For the renormalization of self-consistent approximation schemes we use the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization description. As an example we treat ϕ^4-theory in the phase of unbroken $\mathbb{Z}_2$-symmetry, i.e., we set $m^2 > 0$. Then as well at zero as at finite temperature we have $\varphi = 0$ as the unique solution of the equations of motion. Only the self-consistent propagator has to be determined.

3.1. *Renormalization at $T = 0$*

At $T = 0$ the only difference to the perturbative treatment of the renormalization problem is that we have to apply it to diagrams with lines standing for self-consistent propagators instead for free ones. It is also clear that we can restrict ourselves to the $\{--\}$-part (i.e., the time ordered part) of the real-time contour since in the vacuum case the time-ordered Green's function is identical to the retarded (advanced) one for positive (negative) p^0-components.

The BPHZ renormalization description rests solely on Weinberg's power-counting theorem which is independent of the the special form of propagators. Thus we only have to show that the self-consistent propagators of Φ-derivable approximations belong to the class of functions with the asymptotic behavior $O[(l^2)^{-1}(\ln l^2)^{\beta}]$ for large momenta l^2, where β is a constant. Assuming that this is the case Weinberg's theorem tells us that a connected truncated diagram γ with E external lines has a superficial degree of divergence $\delta(\gamma) = 4 - E$. So due to the Φ-derivable equations of motion the self-consistent self-energy shows an asymptotic behavior like $O[p^2(\ln p^2)^{\beta'}]$. So starting an iteration for the self-consistent propagator with the perturbative propagator (and provided this iteration converges) we can conclude that indeed the propagator is of the usual asymptotic behavior.

The BPHZ renormalization technique aims at the construction of the *integrand* of the renormalized integral without using an intermediate step of regularization. If a diagram is divergent without proper divergent subdiagrams it is sufficient to subtract the Taylor expansion of the integrand with respect to the external momenta up to the order given by the superficial degree of divergence which is in our case $4 - E$, with E denoting the number of external legs.

For renormalization theory it is crucial that the same holds true for diagrams which contain divergences from proper subdiagrams, if the according subdivergences are subtracted first, even if it contains *overlapping* divergences and thus that one needs only *local counterterms local counterterms* to the quantum action which have the same form as that of the classical action but with the infinities lumped into the "bare parameters" rather than the physical ones[14,15].

The described BPHZ-scheme chooses the renormalization point for divergent diagrams at external momenta set to 0. It is clear that by another finite renormalization of the same diagrams we can switch to any renormalization scheme appropriate for the application under consideration. In our case of ϕ^4-theory we choose the on-shell renormalization scheme, which defines the mass parameter m to denote the physical mass of the particles. We use the following on-shell renormalization conditions:

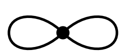
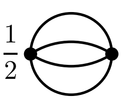
$$\Sigma^{(\mathrm{vac})}(p^2 = m^2) = 0, \quad \partial_{p^2}\Sigma^{(\mathrm{vac})}(p^2 = m^2) = 0, \tag{7}$$

$$\Gamma^{(4,\mathrm{vac})}(s,t,u = 0) = \frac{\lambda}{2}. \tag{8}$$

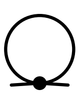
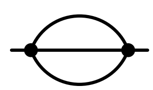
Here s, t, u are the usual Mandelstam variables for two-particle scattering, p is the external momentum of the self-energy and m^2 is the (renormalized) mass of the particles due to the renormalization conditions (7). The second condition defines the wave-function normalization such that the residuum of the propagator at $p^2 = m^2$ is equal to unity.

3.2. *Numerical calculation*

To illustrate the abstract considerations of the previous section we show how to solve the self-consistent Φ-derivable Dyson equation for the ϕ^4 model in the case of unbroken $\mathbb{Z}_2$-symmetry, i.e., for $\varphi = 0$. We take into account the Φ-functional up to three-loop order:

$\Phi =$  $+ \frac{1}{2}$  (9)

For the self-energy we find from Eq. (6)

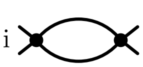
$-\mathrm{i}\Sigma =$  $+$  (10)

The main numerical problem is that it is of course not possible to integrate directly the renormalized integrands of the self-energy diagrams depicted in Eq. (10) because of the on-shell poles of the propagator. Instead we use its Lehmann spectral representation

$$G^{(\mathrm{vac})}(p^2) = \int_0^\infty \frac{\mathrm{d}(m^2)}{\pi} \frac{\operatorname{Im} G^{(\mathrm{vac})}(m^2)}{m^2 - p^2 - \mathrm{i}\eta}, \tag{11}$$

where η denotes a small positive number to be taken to 0^+ in the sense of a weak limit after performing the loop integrals.

First we calculate the one-loop function

$L^{(\mathrm{reg})}(q^2) = \mathrm{i}$  $= \mathrm{i} \int \frac{\mathrm{d}^d l}{(2\pi)^d} G^{(\mathrm{vac})}[(l+q)^2] G^{(\mathrm{vac})}(l^2)$ (12)

which appears as a subdiagram contained in the "sunset diagram" in Eq. (10). Here and in the following we use *dimensional regularization* to give the un-renormalized integrals a definite meaning. At the end of the BPHZ subtraction procedure we can let $d = 4$. The loop function (12) is logarithmically divergent and has to be subtracted such that $L^{(\text{ren})}(q^2 = 0) = 0$ due to the renormalization conditions (7).

Using the spectral representation (11) the *renormalized* loop function can be expressed with help of a kernel $K_1^{(\text{ren})}$:

$$L^{\text{ren}}(q^2) = \int_0^\infty \frac{\mathrm{d}m_1^2}{\pi} \int_0^\infty \frac{\mathrm{d}m_2^2}{\pi} \, K_1^{(\text{ren})}(q^2, m_1^2, m_2^2) \\ \times \operatorname{Im} G^{(\text{vac})}(m_1^2) \operatorname{Im} G^{(\text{vac})}(m_2^2). \tag{13}$$

The renormalized kernel can be calculated analytically with help of standard formulae of perturbation theory (for details see[11]).

Due to the renormalization conditions the tadpole contribution to the self-energy is canceled. For the remaining sunset-diagram we can use the dispersion relation for $L^{(\text{ren})}$ and $G^{(\text{vac})}$ to define a kernel K_2 such that the renormalized self-energy reads

$$\Sigma^{(\text{vac})}(p^2) = \int_{4m^2}^\infty \frac{\mathrm{d}m_3^2}{\pi} \int_0^\infty \frac{\mathrm{d}m_4^2}{\pi} \, K_2^{(\text{ren})}(p^2, m_3^2, m_4^2) \\ \times \operatorname{Im} L^{(\text{ren})}(m_3^2) \operatorname{Im} G^{(\text{vac})}(m_4^2). \tag{14}$$

For the numerical calculation one has to take into account that $\operatorname{Im} G^{(\text{vac})}$ contains the pole contribution $\propto \delta(p^2 - m^2)$ which has to be treated explicitely in both formulae (13) and (14). The remaining integrals over the m_k^2 are relatively smooth finite integrals which can be done with help of a simple adaptive integration algorithm. We used an adaptive Simpson algorithm to solve the equations (13) and (14) iteratively.

As turns out for the vacuum case the main contribution comes from the pole terms such that even for high coupling constants the perturbative and self-consistent result lie on top of each other (see Fig. 2). The reason is that in our on-shell scheme the threshold for the imaginary part of the self-energy is at $9m^2$.

3.3. *Renormalization at $T > 0$*

Now we show that the renormalization at $T > 0$ can be done with the *same temperature independent vacuum counterterms* as were necessary to render the vacuum proper vertex functions finite. Thus in complete analogy to the well-known result of perturbative finite-temperature renormalization theory the renormalized theory is completely defined at $T = 0$. There

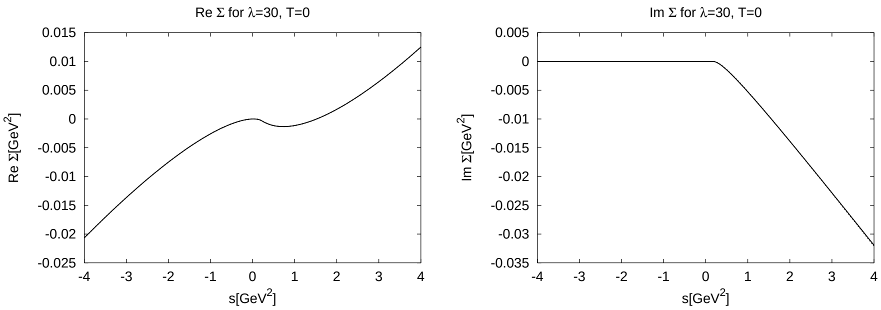


Figure 2. Real (left) and imaginary part (right) of the sunset self- energy. The perturbative and the self-consistent result lie on top of each other due to the large threshold at $s = 9m^2$.

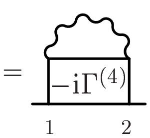
is no ambiguity for "in-medium modifications" of coupling constants from renormalization.

We expand the finite-temperature self-energy around the self-consistent solution of the same Φ-derivable approximation at $T = 0$:

$$\Sigma_{12} = \Sigma_{12}^{(\mathrm{vac})} + \Sigma_{12}^{(0)} + \Sigma_{12}^{(r)}. \tag{15}$$

Here $\Sigma_{12}^{(\mathrm{vac})}$ is the renormalized vacuum self-energy calculated in the previous section. The second and third terms in Eq. (15) contain the in-matter parts of the self-energy. Thereby $\Sigma^{(0)}$ is the part of the self-energy which arises as the *linear part* from a functional power expansion with respect to the Green's function around the vacuum Green's function:

$$-\mathrm{i}\Sigma_{12}^{(0)} = -\mathrm{i}\int_{\mathcal{C}} \mathrm{d}(1'2') \left(\left.\frac{\delta\Sigma_{12}}{\delta G_{1'2'}}\right|_{T=0} G_{1'2'}^{(\mathrm{mat})} \right) = \tag{16}$$



The wavy line stands for the "matter part" of the Green's function $G^{(\mathrm{mat})} = G - G^{(\mathrm{vac})}$. As we shall see below herein we have to understand only the *diagonal part* of the vacuum propagator within the momentum-space matrix formalism. The four-point kernel $\Gamma^{(4)}$ is a four-point function represented by a particular set of self-energy subdiagrams consisting of pure vacuum lines, defined by

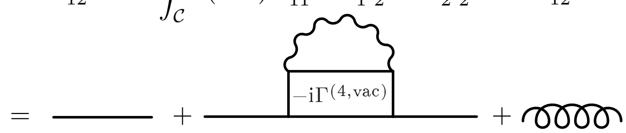
$$-\mathrm{i}\Gamma^{(4)}_{12,1'2'} = -\left.\frac{\delta\Sigma_{12}}{\delta G_{1'2'}}\right|_{T=0} = -2\mathrm{i}\left.\frac{\delta^2\Phi}{\delta G_{12}\delta G_{1'2'}}\right|_{T=0}. \tag{17}$$

Its "diagonal part", i.e., with all time arguments placed on one side of the real-time contour, defines a vacuum renormalization part which is of superficial degree of divergence 0. Thus we can conclude that the *diagonal part* of

$G^{(\mathrm{mat})}$ is of momentum power -4, so that closing $\Gamma^{(4)}$ with a wavy $G^{(\mathrm{mat})}$-line yields another logarithmic divergence, even when the pure vacuum part $\Gamma^{(4)}$ is renormalized. On the other hand the off-diagonal parts of $G^{(\mathrm{mat})}$ contain θ-functions and Bose-Einstein-distribution factors which lead to convergent temperature dependent integrals which we are not allowed to subtract. Thus from Weinberg's power-counting theorem we can conclude that the divergent part of $\Sigma^{(0)}$, in the following called $\Sigma^{(0,\mathrm{div})}$, accounts for all terms of momentum power 0 and consequently $\Sigma^{(r)}$ is of divergence degree -2 and thus finite after subtracting vacuum subdivergences.

So we are left with the task to renormalize the last loop integral from closing the $\Gamma^{(4)}$-diagram with a $G^{(\mathrm{mat})}$-line. For this purpose due to our discussion above we have to split the full propagator as follows

$$\mathrm{i}G_{12} = \mathrm{i}G_{12}^{(\mathrm{vac})} + \mathrm{i}\int_{\mathcal{C}} \mathrm{d}(1'2') G_{11'}^{(\mathrm{vac})} \Sigma_{1'2'}^{(0,\mathrm{div})} G_{2'2}^{(\mathrm{vac})} + \mathrm{i}G_{12}^{(r)} \tag{18}$$

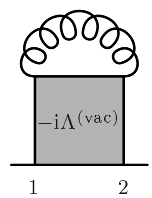


Using Eq. (18) shows that $\Sigma^{(0,\mathrm{div})}$, represented by the second diagram in (18), fulfills the equation of motion

$$\Sigma_{12}^{(0,\mathrm{div})} = \int_{\mathcal{C}} \mathrm{d}(1'2') \Gamma_{12,1'2'}^{(4,\mathrm{vac})} \left(\int_{\mathcal{C}} \mathrm{d}(1''2'') G_{1'1''}^{(\mathrm{vac})} \Sigma_{1''2''}^{(0,\mathrm{div})} G_{2''2'}^{(\mathrm{vac})} + G_{1'2'}^{(r)} \right) \tag{19}$$

which is *linear* in $G^{(r)}$. Thus Eq. (19) is solved by the ansatz

$$-\mathrm{i}\Sigma_{12}^{(0,\mathrm{div})} = \int_{\mathcal{C}} \mathrm{d}(1'2') \Lambda_{12,1'2'} G_{1'2'}^{(r)} = \tag{20}$$



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where the vacuum four-point function $\Lambda^{(\mathrm{vac})}$ fulfills the *Bethe-Salpeter equation*

$$\Lambda_{12,1'2'}^{(\mathrm{vac})} = \Gamma_{12,1'2'}^{(4,\mathrm{vac})} + \mathrm{i}\int_{\mathcal{C}} \mathrm{d}(3456) \Gamma_{12,34}^{(4,\mathrm{vac})} G_{35}^{(\mathrm{vac})} G_{46}^{(\mathrm{vac})} \Lambda_{56,1'2'}. \tag{21}$$

Once this logarithmically divergent vacuum subdivergence is renormalized also $\Sigma^{(0,\mathrm{div})}$ is finite since $G^{(r)}$ is falling off with momentum power -6.

For the renormalization of the four-point function we note that the momentum-space version of (21) reads

$$\begin{aligned} \Lambda^{(\mathrm{vac})}(p,q) &= \Gamma^{(4,\mathrm{vac})}(p,q) + \mathrm{i}\int \tfrac{\mathrm{d}^d l}{(2\pi)^d} \Gamma^{(4,\mathrm{vac})}(p,l) [G^{(\mathrm{vac})}(l)]^2 \Lambda^{(\mathrm{vac})}(l,q) \\ &= \Gamma^{(4,\mathrm{vac})}(p,q) + \mathrm{i}\int \tfrac{\mathrm{d}^d l}{(2\pi)^d} \Lambda^{(\mathrm{vac})}(p,l) [G^{(\mathrm{vac})}(l)]^2 \Gamma^{(4,\mathrm{vac})}(l,q) \end{aligned} \tag{22}$$

To renormalize this equation, a detailed BPHZ-analysis uses the fact that due to the 2PI-feature of the Φ functional $\Gamma^{(4,\mathrm{vac})}_{12,34}$ is 2PI relative to any cut which separates the space-time point pairs (12) and (34). Thus there is no "BPHZ-box" cutting through this Bethe-Salpeter kernel. Thus we can do the subtractions at the upper and the lower end of any subdivergence with the renormalized BS-kernel leading to the *renormalized BS-equation*

$$\begin{aligned}\Lambda^{(\mathrm{ren})}(p,q) = \Gamma^{(4,\mathrm{ren})}(p,q) \\ &+\mathrm{i}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}[\Gamma^{(4,\mathrm{ren})}(p,l)-\Gamma^{(4,\mathrm{vac})}(0,l)][G^{(\mathrm{vac})}(l)]^2\Lambda^{(\mathrm{ren})} \\ &+\mathrm{i}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\Lambda^{(\mathrm{ren})}(0,l)[G^{(\mathrm{vac})}(l)]^2[\Gamma^{(4,\mathrm{ren})}(l,q)-\Gamma^{(4,\mathrm{ren})}(l,0)].\end{aligned} \tag{23}$$

The renormalization of $\Gamma^{(4,\mathrm{vac})}$ itself is straight forward, since it is given by a finite set of vacuum diagrams which can be renormalized by the same BPHZ-scheme as the perturbative ones.

For the practical calculation of the self-energy part $\Sigma^{(0)}$ we need only the $\Lambda^{(\mathrm{ren})}(0,q)$. Indeed using Eqs. (18) and (20) we find

$$\begin{aligned}\Sigma^{(0)}(p) &= \Sigma^{(0)}(p)-\Sigma^{(0)}(p)+\Sigma^{(0)}(p) \\ &= \int \frac{\mathrm{d}^4 l}{(2\pi)^4}[\Gamma^{(4,\mathrm{ren})}(p,l)-\Gamma^{(4,\mathrm{ren})}(0,l)]G^{(\mathrm{matter})}(l) \\ &+\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\Lambda^{(\mathrm{ren})}(0,l)G^{(r)}(l)\end{aligned} \tag{24}$$

An example solution for the equations and the comparison with the perturbative approximation is shown in Fig. 3.

While at $T=0$ both, the perturbative and the self-consistent solution, show the three-particle threshold at $\sqrt{s}=3m$ at finite temperature the spectral width smoothes out all threshold structures in the self-consistent solution. The growing high-energy tail is related to the decay of a virtual particle to three particles. At finite temperature as an additional effect a low-energy plateau in $\mathrm{Im}\,\Sigma^R$ emerges from in-medium scattering processes of real particles.

The comparison of the self-consistent solutions with the perturbative approximation shows counterbalancing effects of self-consistency: The finite spectral width, contained in the self-consistent propagator leads to a further broadening of the width and a smoothing of the structure as a function of energy. This is counterbalanced by the behavior of the real part of the self-energy which essentially shifts the in-medium mass upwards. This reduces the available phase space for real processes. With increasing coupling strength λ a nearly linear behavior of $\mathrm{Im}\,\Sigma^R$ with p_0 results, implying a nearly constant damping width, given by $-\mathrm{Im}\,\Sigma^R/p_0$.

While the tadpole contribution always shifts the mass upwards, at higher couplings and temperature also the real part of the sunset diagram

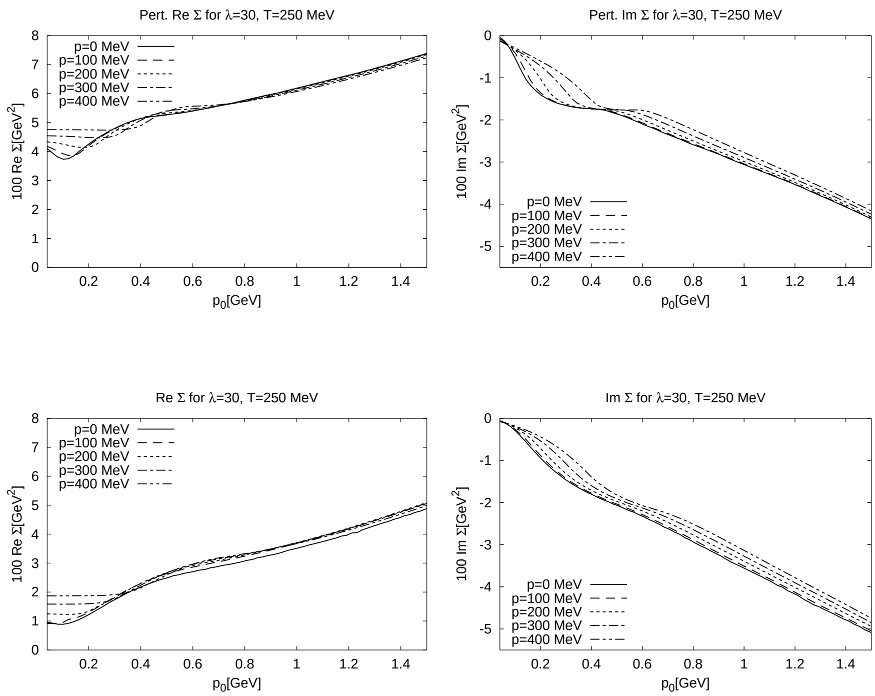


Figure 3. Real (left) and imaginary part (right) of the perturbative (top) and the self-consistent self-energy for $\lambda = 30$, $m = 140$MeV and $T = 250$MeV. Note that the self-energies are multiplied with an factor 100 in these plots!

becomes significant which can lead to a net down-shift of the mass again. This can be seen for the parameter set used for the Fig. 3.

4. Global symmetries

If the classical action underlying a quantum field theory is invariant under the linear operation of a Lie group G, where the group elements are independent of the space time argument one speaks of a *global symmetry of the classical action*. Then it can be shown that for each linearly independent generator $t^a \in \mathcal{L}G$ there exists a conserved charge Q^a which in turn builds a basis of the Lie algebra $\mathcal{L}G$ as a subalgebra of the canonical Poisson algebra of the fields.

It is a well-known theorem in perturbative quantum field theory that the same holds true for the quantized theory, i.e., the effective action is symmetric under the same symmetry group as the classical action provided that no quantum anomaly destroys the symmetry or a part of it. The n-

point Green's functions fulfill certain constraints due to the symmetry which we shall call the *Ward-Takahashi identities (WTIs)* of the symmetry.

For Φ-derivable approximations in general the WTIs of the symmetry are violated for the two-point and higher vertex functions although the expectation values of the corresponding Noether currents are exactly conserved. This can be traced back to a violation of crossing symmetry within the self-consistent propagator due to partial resummation of the self-energy insertions by means of the Dyson equation of motion. We shall further show that the WTIs are fulfilled for vertex functions, which we shall call *external*. These functions are defined from a non-perturbative effective action which is uniquely determined by the underlying Φ-derivable approximation.

We study the symmetry properties of the generating functional (2) of a scalar $\mathrm{O}(N)$-symmetric quantum field theory. An infinitesimal $\mathrm{O}(N)$-transformation reads $\delta\phi_1^j = \mathrm{i}\delta\chi_1^a (\tau^a)^j{}_{j'}\phi_1^{j'}$. Making use of the assumed invariance of the classical action under $\mathrm{O}(N)$-transformations one finds by standard path-integral analysis[13] that $\Gamma[\varphi, G]$ is an $\mathrm{O}(N)$-scalar functional when ϕ is transformed as a vector and G as a second-rank tensor:

$$\int_{\mathcal{C}} \mathrm{d}(1) \frac{\delta\Gamma}{\delta\phi_1^j} (\tau^a)^j{}_k \phi_1^k + \int_{\mathcal{C}} \mathrm{d}(12) \frac{\delta\Gamma}{\delta G_{12}^{jk}} [(\tau^a)^j{}_{j'} G_{12}^{j'k} + (\tau^a)^k{}_{k'} G_{12}^{jk'}] = 0. \quad (25)$$

Now only for the exact functional the self-consistent Green's function G is identical with the exact one. Thus in general only for the exact case the self-energy and higher vertex functions fulfill all WTIs of the usual 1PI action.

Generally for a Φ-derivable approximation this equality of vertex functions does not hold true any longer. For such approximations in general the WTIs are violated in loop orders higher than that taken into account for the approximation of the functional. The reason is that the solution of the equations of motion are equivalent to a certain partial resummation of the perturbation series which is not crossing symmetric in the inner structure of the diagrams.

To recover the crossing symmetry we define an effective action functional from our approximated 2PI-action as

$$\tilde{\Gamma}[\varphi] = \Gamma[\varphi, \tilde{G}[\varphi]] \text{ with } \left. \frac{\delta\Gamma[\varphi, G]}{\delta G} \right|_{G=\tilde{G}[\varphi]} = 0. \quad (26)$$

The stationary point of this action functional defines the mean field and propagator of the Φ-derivable approximation since

$$\frac{\delta\tilde{\Gamma}}{\delta\varphi} = \left(\frac{\delta\Gamma[\varphi, G]}{\delta\varphi} + \int_{\mathcal{C}} \mathrm{d}(12) \frac{\delta\Gamma[\varphi, G]}{\delta G_{12}^{jk}} \frac{\delta\tilde{G}_{12}^{jk}[\varphi]}{\delta\varphi} \right)_{G=\tilde{G}[\varphi]}. \quad (27)$$

From (26) this yields

$$\frac{\delta\tilde{\Gamma}}{\delta\varphi} = \left(\frac{\delta\Gamma[\varphi, G]}{\delta\varphi}\right)_{G=\tilde{G}[\varphi]}. \tag{28}$$

Thus the stationary point $\tilde{\varphi}$ of $\tilde{\Gamma}$ is identical to the mean field of the Φ-derivable approximation and together with (26) this means that $\tilde{G}[\tilde{\varphi}]$ is the solution of the Dyson equation from the same Φ-functional.

For the exact functional $\tilde{\Gamma}$ is the usual effective quantum action. Thus for a Φ-derivable approximation $\tilde{\Gamma}$ defines a non-perturbative approximation to this functional and can be used to derive approximations for the proper vertex functions:

$$(\tilde{\Gamma}^{(n)})_{12\ldots}^{jk\ldots} = \frac{\delta^n\tilde{\Gamma}[\varphi]}{\delta\varphi_1^j\delta\varphi_2^k\cdots}. \tag{29}$$

Especially $\tilde{\Gamma}^{(2)}$ is an approximation for the inverse propagator, which we call the *external propagator* to be distinguished from the self-consistent propagator. This external propagator fulfills the usual WTI but is not identical with the self-consistent or *internal* propagator.

The vertex functions (29) are by definition crossing symmetric. Now the symmetry property (25) by construction holds also true for the approximated Φ-functional and thus from (26) we see that $\tilde{\Gamma}$ is an $\mathrm{O}(N)$-scalar functional:

$$\int_{\mathcal{C}} \mathrm{d}(1) \frac{\delta\tilde{\Gamma}[\varphi]}{\delta\varphi_1^j}(\tau^a)^j{}_{j'}\varphi_1^{j'} = 0. \tag{30}$$

This symmetry property contains all WTIs for the external vertex functions. Especially for the external propagator, defined by

$$(G_{\text{ext}}^{-1})_{1j,2k} = \left.\frac{\delta^2\tilde{\Gamma}[\varphi]}{\delta\varphi_1^j\delta\varphi_2^k}\right|_{\varphi=\tilde{\varphi}}, \tag{31}$$

we find by taking the functional derivative of (30)

$$\int_{\mathcal{C}} \mathrm{d}(1)(G_{\text{ext}}^{-1})_{1j,2k}(\tau^a)^j{}_{j'}\tilde{\varphi}_1^{j'} = 0. \tag{32}$$

For a translationally invariant state the Fourier transform of (32) with respect to $(x_1 - x_2)$ reads

$$(G_{\text{ext}}^{-1})_{jk}(p=0)(\tau^a)^j{}_{j'}\varphi^{j'} = -(M^2)_{jk}(\tau^a)^j{}_{j'}\tilde{\varphi}^{j'} = 0. \tag{33}$$

It is clear that in this situation $\tilde{\varphi}$ is a constant due to translation invariance. If it is not 0 the symmetry is spontaneously broken. Since $(M^2)_{jk}$ is the mass matrix of the particles described as the excitations of the field around the mean field $\tilde{\varphi}$ Eq. (33) tells us that the $N-1$ fields perpendicular to the direction given by the arbitrarily chosen solution $\tilde{\varphi}$ are massless, the *Nambu-Goldstone bosons* of the symmetry.

To calculate the external propagator we apply (26) in (31) to obtain

$$(G_{\text{ext}}^{-1})_{1j,2k} = \left[\frac{\delta^2\Gamma[\varphi,G]}{\delta\varphi_1^j\delta\varphi_2^k} + \int_{\mathcal{C}} \mathrm{d}(3'4')\frac{\delta^2\Gamma[\varphi,G]}{\varphi_1^j\delta G_{3'4'}^{j'k'}}\frac{\delta\tilde{G}_{3'4'}^{j'k'}}{\delta\varphi_2^k}\right]_{\varphi=\tilde{\varphi},\, G=\tilde{G}[\tilde{\varphi}]}. \tag{34}$$

Taking the derivative of the identity

$$\int_{\mathcal{C}} \mathrm{d}(2')(\tilde{G}^{-1})_{1j,2'k'}\tilde{G}_{2'2}^{k'k} = \delta_{12}^{(d)}\delta_j^k \tag{35}$$

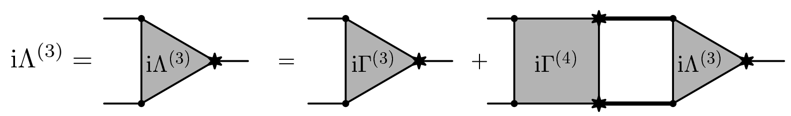
with respect to the field we get

$$\int_{\mathcal{C}} \mathrm{d}(1'2')\left[\frac{\delta\tilde{G}^{-1}}{\delta\varphi_3^l}\tilde{G}_{2'2}^{k'k} + (\tilde{G}^{-1})_{1j,2'k'}\frac{\delta G_{2'2}^{k'k}}{\delta\varphi_3^l}\right] = 0. \tag{36}$$

With help of (6) the three-point function

$$\Lambda_{1j,2k;3l}^{(3)} = \frac{\delta\tilde{G}_{1j,2k}^{-1}}{\delta\varphi_3^l} \tag{37}$$

can be expressed as the solution of the BS-equation

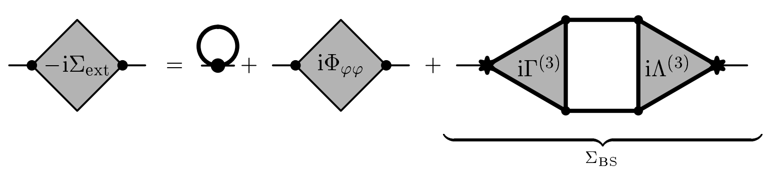


(38)

while its kernels are determined by the Φ-functional:

$$\begin{aligned}
\Gamma_{1j,2k;3l}^{(3)} &= \left[\frac{\delta^3 S[\varphi]}{\delta\varphi_1^j\delta\varphi_2^k\delta\varphi_3^l} - 2\mathrm{i}\frac{\delta^2\Phi[\varphi,G]}{\delta G_{12}^{jk}\delta\varphi_3^l}\right]_{G=\tilde{G}[\tilde{\varphi}],\, \varphi=\tilde{\varphi}} \\
\Gamma_{1j,2k;3l,4m}^{(4)} &= -2\left[\frac{\delta^2\Phi[\varphi,G]}{\delta G_{12}^{jk}\delta G_{34}^{lm}}\right]_{G=\tilde{G}[\tilde{\varphi}],\, \varphi=\tilde{\varphi}}
\end{aligned} \tag{39}$$

With these definitions (34) can be written as



(40)

It is clear that both, the BS-equation (38) and Eq. (40), have to be renormalized. This is done again with help of the above explained BPHZ techniques. Again all counterterms are independent of the temperature and consistent with those needed to renormalize the underlying self-consistent equations of motion.

As an example we show results for the lowest order approximation which is the Hartree approximation for the self-consistent self-energy, leading to a constant effective mass shown in Fig. 4. Clearly Goldstone's theorem

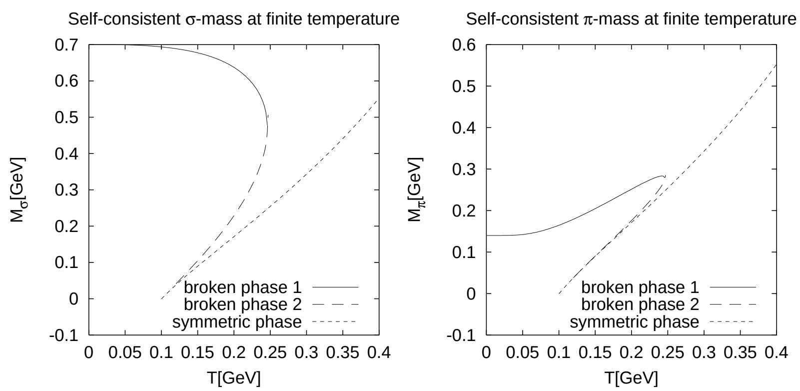


Figure 4. The solutions for the Hartree approximation

is violated, since the pion mass is different from 0 in both broken phases. In this case the external self-energy is obtained by a bubble resummation, corresponding to a *Random phase approximation (RPA)*. Since at the same time it is the second functional derivative of the effective 1PI-action functional it provides a *stability criterion* for the Hartree solution, which is of course only a stable solution, i.e., a minimum of the effective potential, if the mass matrix is positive semidefinite. It turns out that the solution, denoted as "broken phase 2" in Fig. 4 is *unstable*. In Fig. 5 the effective mass obtained from the *external propagator*, is shown. Indeed Goldstone's theorem is fulfilled, as to be expected from our analysis.

References

1. J. Luttinger, J. Ward, Phys. Rev. **118** (1960) 1417
2. T. D. Lee, C. N. Yang, Phys. Rev. **117** (1961) 22
3. G. Baym, L. Kadanoff, Phys. Rev. **124** (1961) 287
4. G. Baym, Phys. Rev. **127** (1962) 1391
5. M. Cornwall, R. Jackiw, E. Tomboulis, Phys. Rev. D **10** (1974) 2428
6. J. Schwinger, J. Math. Phys **2** (1961) 407
7. L. Keldysh, Zh. Eksp. Teor. Fiz. **47** (1964) 1515, [Sov. Phys JETP **20** 1965 1018]
8. A. F. Bielajew, Nuclear Physics **A404** (1983) 428

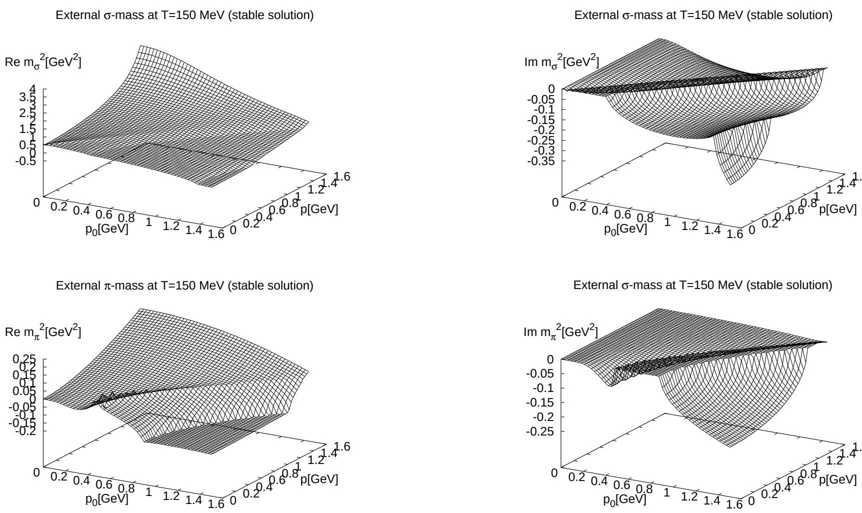


Figure 5. The effective external masses at a temperature of 150MeV. The effective external π-mass indeed vanishes at $p_0 = \vec{p} = 0$ as predicted from Goldstone's theorem. The spectral function of the σ-meson shows that at high temperatures its strength becomes more peaked and the maximum shifted to lower momenta than at $T = 0$.

9. A. F. Bielajew, B. D. Serot, Ann. Phys. (NY) **156** (1984) 215
10. H. van Hees, J. Knoll, Phys. Rev. D **65** (2002) 025010
11. H. van Hees, J. Knoll, Phys. Rev. D **65** (2002) 105005
12. G. Baym, G. Grinstein, Phys. Rev. D **15** (1977) 2897
13. H. van Hees, J. Knoll, Phys. Rev. D **66** (2002) 025028
14. N. N. Bogoliubov, O. S. Parasiuk, Acta Math. **97** (1957) 227
15. W. Zimmermann, Commun. Math. Phys. **15** (1969) 208